

Calculation of PDH Error Signal Slope in the limit of sidebands being inside the linewidth

This notebook shows that the slope of PDH Error Signal for Transmission is $\frac{32\alpha\mathcal{F}}{\pi\lambda} F[\omega + \Omega] F[\omega - \Omega]$. In comparison to Black's result for slope in reflection out of the linewidth, there is an extra factor of $\frac{4\alpha}{\pi} F[\omega + \Omega] F[\omega - \Omega]$

Coefficients for Reflection/Transmission from a cavity as a function of frequency and length

$$\mathbf{FRef1}[\omega_ , L_] := \frac{r1 - r2 (r1^2 + t1^2) e^{-\frac{2i\omega L}{c}}}{1 - r1 r2 e^{-\frac{2i\omega L}{c}}}$$

$$\mathbf{FTrans}[\omega_ , L_] := \frac{t1 t2 e^{-\frac{i\omega L}{c}}}{1 - r1 r2 e^{-\frac{2i\omega L}{c}}}$$

$$\mathbf{F}[\omega_ , L_] = \mathbf{FTrans}[\omega, L];$$

The total power on the PD after demodulation at sideband frequency Ω

$$\mathbf{PowerMeasured}[\omega_ , L_ , \Omega_ , \phi_] :=$$

$$Pc \text{Abs}[F[\omega, L]]^2 + Ps \left(\text{Abs}[F[\omega + \Omega, L]]^2 + \text{Abs}[F[\omega - \Omega, L]]^2 \right) +$$

$$2 \text{Sqrt}[Pc Ps] \left(\text{Re}[F[\omega, L] F[\omega + \Omega, L]^* - F[\omega, L]^* F[\omega - \Omega, L]] \text{Cos}[\phi] + \right.$$

$$\left. \text{Im}[F[\omega, L] F[\omega + \Omega, L]^* - F[\omega, L]^* F[\omega - \Omega, L]] \text{Sin}[\phi] \right)$$

(*Here ϕ is the phase of the demodulation LO, at the frequency of the sidebands*)

Thus, only considering the last two terms, (without the $2 \text{Sqrt}[Pc Ps]$)

```
ErrSig[ω_, L_, Ω_, φ_] :=
  FullSimplify[ComplexExpand[Re[F[ω, L] F[ω + Ω, L]* - F[ω, L]* F[ω - Ω, L]] Cos[φ] +
    Im[F[ω, L] F[ω + Ω, L]* - F[ω, L]* F[ω - Ω, L]] Sin[φ]]];
ErrSig[ω, L, Ω, φ] // TraditionalForm
```

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Beep[]
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$$\begin{aligned} & \left(r_1 r_2 t_1^2 t_2^2 \left(r_1 r_2 \left(-4 r_1 r_2 \sin\left(\frac{2L\omega}{c}\right) \sin\left(\frac{L\Omega}{c}\right) \cos\left(\frac{2L\Omega}{c} + \phi\right) - \sin(\phi) \sin\left(\frac{L(4\omega + \Omega)}{c}\right) + \right. \right. \right. \\ & \quad \left. \left. \left. 2 \sin\left(\frac{4L\omega}{c}\right) \sin\left(\frac{L\Omega}{c} + \phi\right) - \cos(\phi) \cos\left(\frac{L(4\omega + \Omega)}{c}\right) + \cos\left(\frac{c\phi + 4L\omega - L\Omega}{c}\right) \right) + \right. \right. \\ & \quad \left. \left. \cos\left(\frac{c\phi - 2L\omega - 3L\Omega}{c}\right) - \cos\left(\frac{c\phi + 2L\omega - 3L\Omega}{c}\right) + \cos\left(\frac{c\phi + 2L\omega - L\Omega}{c}\right) - \cos\left(\phi - \frac{L(2\omega + \Omega)}{c}\right) \right) \right) / \\ & \left(\left(-2 r_1 r_2 \cos\left(\frac{2L\omega}{c}\right) + r_1^2 r_2^2 + 1 \right) \left(-2 r_1 r_2 \cos\left(\frac{2L(\omega - \Omega)}{c}\right) + r_1^2 r_2^2 + 1 \right) \right. \\ & \quad \left. \left(-2 r_1 r_2 \cos\left(\frac{2L(\omega + \Omega)}{c}\right) + r_1^2 r_2^2 + 1 \right) \right) \end{aligned}$$

Differentiate the Error Signal wrt L to get the linear term in its Taylor series

```
PDHLinSignal[ω_, L_, Ω_, φ_] = D[ErrSig[ω, L, Ω, φ], L];
```

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Beep[]
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Error Signal Slope

Compute the slope at a length on which the cavity is on resonance, ie $L = \frac{2\pi c M}{\omega}$

```
ErrSigSlope[ω_, L_, Ω_, φ_] := FullSimplify[PDHLinSignal[ω, 2 π c  $\frac{M}{\omega}$ , Ω, φ],
```

```
  Assumptions → Element[M, Integers]] /. {M →  $\frac{\omega L}{2 \pi c}$ };
```

```
ErrSigSlope[ω, L, Ω, φ] // TraditionalForm
```

```
Beep[]
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$$\begin{aligned} & - \left(\left(8 r_1 r_2 t_1^2 t_2^2 \omega \sin\left(\frac{L\Omega}{c}\right) \left(r_1 r_2 \left(r_1 r_2 \cos\left(\frac{2L\Omega}{c} + \phi\right) - 2 \cos(\phi) \right) + \cos\left(\phi - \frac{2L\Omega}{c}\right) \right) \right) / \right. \\ & \quad \left. \left(c (r_1 r_2 - 1)^2 \left(-2 r_1 r_2 \cos\left(\frac{2L\Omega}{c}\right) + r_1^2 r_2^2 + 1 \right)^2 \right) \right) \end{aligned}$$

(*2 π M c = L ω => M = $\frac{L\omega}{2 \pi c}$ *)

Find the demod phase that maximizes the slope

```
SolutionForDemodAngle[L_, Ω_] =
  FullSimplify[Solve[TrigExpand[D[ErrSigSlope[ω, L, Ω, φ], φ]] == 0, φ],
    Assumptions → {L, c, r1, r2, Ω} > 0];
SolutionForDemodAngle[L, Ω] // TraditionalForm
Beep[]
```

$$\left\{ \left\{ \phi \rightarrow \text{ConditionalExpression} \left[\tan^{-1} \left(\frac{\sin\left(\frac{L\Omega}{c}\right) \left((r1^2 r2^2 + 1) \cos\left(\frac{2L\Omega}{c}\right) - 2 r1 r2 \right)}{\sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}} \right), \right. \right. \\ \left. \left. \frac{(r1^2 r2^2 - 1) \sin\left(\frac{L\Omega}{c}\right) \sin\left(\frac{2L\Omega}{c}\right)}{\sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}} + 2 \pi c_1, c_1 \in \mathbb{Z} \right\} \right\}, \\ \left\{ \phi \rightarrow \text{ConditionalExpression} \left[\tan^{-1} \left(\frac{\sin\left(\frac{L\Omega}{c}\right) \left((r1^2 r2^2 + 1) \cos\left(\frac{2L\Omega}{c}\right) - 2 r1 r2 \right)}{\sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}} \right), \right. \right. \\ \left. \left. \frac{(1 - r1^2 r2^2) \sin\left(\frac{L\Omega}{c}\right) \sin\left(\frac{2L\Omega}{c}\right)}{\sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}} + 2 \pi c_1, c_1 \in \mathbb{Z} \right\} \right\}$$

Substitute that angle to get the maximum slope

```
MaxErrSignalSlope[ω_, L_, Ω_] =
  Normal[FullSimplify[ErrSigSlope[ω, L, Ω, φ /. SolutionForDemodAngle[L, Ω]]]];
MaxErrSignalSlope[ω, L, Ω] // TraditionalForm
Beep[]
```

$$\left\{ \frac{8 r1 r2 t1^2 t2^2 \omega \sin^2\left(\frac{L\Omega}{c}\right)}{c (r1 r2 - 1)^2 \sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}}, \right. \\ \left. - \frac{8 r1 r2 t1^2 t2^2 \omega \sin^2\left(\frac{L\Omega}{c}\right)}{c (r1 r2 - 1)^2 \sqrt{\sin^2\left(\frac{L\Omega}{c}\right) \left(-2 r1 r2 \cos\left(\frac{2L\Omega}{c}\right) + r1^2 r2^2 + 1 \right)^2}} \right\}$$

Absolute value makes both solutions the same, so focus on only one from now on.

Now Simplify by finding familiar factors in this slope

$F[\omega+\Omega]$ $F[\omega-\Omega]$ appears in the slope

`RemoveAbs[x_] := x`

`FsInSlope[ω , L, Ω] = FullSimplify[ComplexExpand[F[$\omega + \Omega$, $\frac{2 \pi M c}{\omega}$] F[$\omega - \Omega$, $\frac{2 \pi M c}{\omega}$]],`
`Assumptions \rightarrow Element[M, Integers]] /. {M \rightarrow $\frac{\omega L}{2 \pi c}$ };`

`SimpleSlope[ω _, L_, Ω _, FFStar_] = FullSimplify[Abs[MaxErrSignalSlope[ω , L, Ω][[1]]],`
`FFStar == Abs[FsInSlope[ω , L, Ω]]] /. Abs \rightarrow RemoveAbs`

$$\frac{8 \text{FFStar } r1 \text{ } r2 \text{ } \omega \text{ Sin}\left[\frac{L \Omega}{c}\right]}{c (-1 + r1 \text{ } r2)^2}$$

There appears to be \mathcal{F}^2 in this expression, but the Sin() part cancels one.

For the sin() part, make the approximation that the sideband frequency is much smaller than the fsr if the sideband is inside the linewidth, ie high finesse.

$$\Omega = x \frac{2c}{L} = \alpha \gamma = \alpha \frac{2c}{L} \frac{1}{\mathcal{F}}$$

And substitute for Finesse

`Simplify[Solve[$\mathcal{F} == \frac{\text{Sqrt}[r1r2] \pi}{(1 - r1r2)}$, r1r2]]`

$$\left\{ \left\{ r1r2 \rightarrow \frac{\pi^2 + 2 \mathcal{F}^2 - \pi \sqrt{\pi^2 + 4 \mathcal{F}^2}}{2 \mathcal{F}^2} \right\}, \left\{ r1r2 \rightarrow \frac{\pi^2 + 2 \mathcal{F}^2 + \pi \sqrt{\pi^2 + 4 \mathcal{F}^2}}{2 \mathcal{F}^2} \right\} \right\}$$

`HighFinesseSlope[ω _, L_, Ω _, FFStar_, α _, \mathcal{F} _] = Simplify[`

`Simplify[Normal[Series[SimpleSlope[ω , L, $\frac{2 c x}{L}$, FFStar], {x, 0, 1}]]] /. x \rightarrow $\frac{\alpha}{\mathcal{F}}$] /.`

$$r1 \rightarrow \frac{\pi^2 + 2 \mathcal{F}^2 - \pi \sqrt{\pi^2 + 4 \mathcal{F}^2}}{2 r2 \mathcal{F}^2} \quad // \text{TraditionalForm}$$

$$\frac{16 \text{FFStar } \mathcal{F} \alpha \omega}{c \pi^2}$$

Thus the slope of PDH Error Signal for Transmission is

$\frac{32\alpha\mathcal{F}}{\pi\lambda} F[\omega + \Omega] F[\omega - \Omega]$. In comparison to Black's result for

slope in reflection out of the linewidth, there is an extra factor of

$\frac{4\alpha}{\pi} F[\omega + \Omega] F[\omega - \Omega]$

$$c = 3 \times 10^8;$$

$$\omega = \frac{2\pi c}{1064 \times 10^{-9}};$$

$$L = 106.4 \times 10^{-6};$$

$$t1 = \text{Sqrt}[50 \times 10^{-6}];$$

$$t2 = \text{Sqrt}[50 \times 10^{-6}];$$

$$r1 = \text{Sqrt}[1 - 50 \times 10^{-6}];$$

$$r2 = \text{Sqrt}[1 - 50 \times 10^{-6}];$$

`MaxErrSignalSlope[ω , L, 105][[1]]`

$$6.70172 \times 10^8$$

`Plot[MaxErrSignalSlope[ω , L, Ω][[1]], { Ω , -1013, 1013}`

