A mathematical relation to make the right curlicue

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I was making a curlicue origami one fine snow-driven Sunday. The design, made by Assia Brill can be seen at `https://www.youtube.com/watch?v=Kt4r7MAG4Lg` and `https://www.youtube.com/watch?v=LZ0Ik0zFUL4`.

After I had made a model from an A4 sheet of paper (Fig. 1), I decided I wanted to make a real version. Thus I acquired a big enough sheet of paper. But soon I was presented by a question : how big should I cut my sheet? The video gives a rough measurement for the curlicues she made, but I had no idea of what I should use given how big a curlicue I want. I thought about it and saw how simple the math was, and it was not obvious to me at first. Since I wanted to finish that origami then, I didnt sit down to calculate. Instead I just made the strip as long as my dining table, (I didnt have a flat surface to draw longer than that in my carpetted home). What I made thus can be seen in Fig. 2

That didn't really give me a peace of mind, so I later sat down to derive that relation between the dimensions of the paper and the number of layers when folding this type of curlicue. I am going to lay out that calculation and its results in this report.

For the purpose of this calculation, I have drawn what the creases look like (in green) in Fig 3. It was not needed to differentiate between mountain and

Figure 1: Curlicue made out of an A4 sheet, the colours are just the other side of the paper.

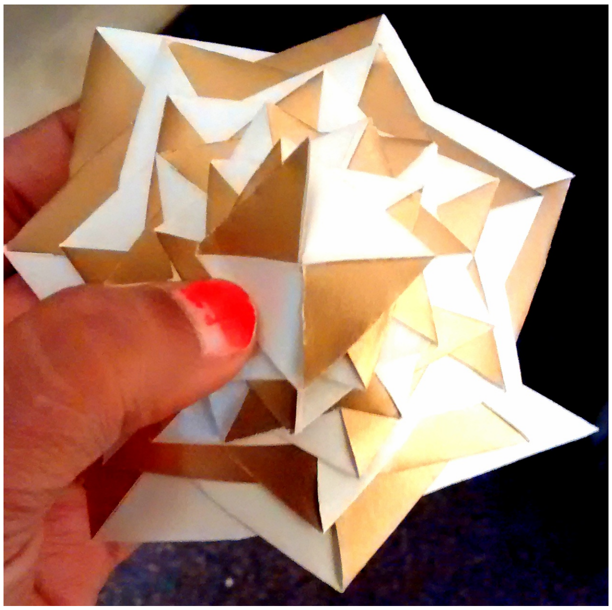

Figure 2: Curlicue made out of a longer, almost 2.5 meter long strip.

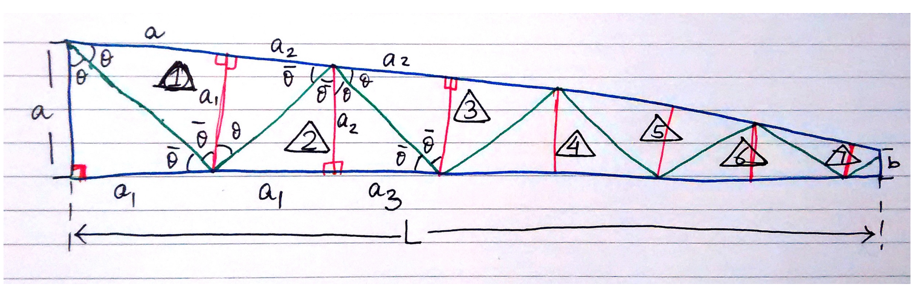

Figure 3: Dimensions of the strip used to make the curlicue, important quantities are a,b,L and θ. Green lines are fold lines, the numbers written in a triangle are the index of the 'big triangles'.

valley folds for this calculation, but it should be straight forward. If one tries to fold this strip, its easy to find that the number of layers in the final curlicue is 1/4 times the number of 'big triangles' - the triangles formed by two green and one blue side, see Fig 3

The length of the strip is measured along the perpendicular edge. Length of nth big triangle = $a_{n-1} + a_{n-2}$ From Fig 3, one can write down the length for 2n big triangles

$$L(2n) = a_1 + a_1 + a_3 + a_3 + a_5 + ... + a_{2n-1} + a_{2n+1} \tag{1}$$
$$L(2n) = 2a_1 + 2a_3 + 2a_5 + ... + 2a_{2n-1} + a_{2n+1} \tag{2}$$

Now,

$$a_n = a_{n-1} \tan\theta \tag{3}$$
$$a_n = a \tan^n \theta \tag{4}$$

Thus,

$$L(2n) = 2(a\tan\theta + a\tan^3\theta + ... + a\tan^{2n-1}\theta) + a\tan^{2n+1}\theta \tag{5}$$
$$= 2a\tan\theta(1 + \tan^2\theta + ... + \tan^{2(n-1)}\theta) + a\tan^{2n+1}\theta \tag{6}$$
$$= 2a\tan\theta \frac{(1 - (\tan^2\theta)^n)}{1 - \tan^2\theta} + a\tan^{2n+1}\theta \tag{7}$$

Now, assuming we want m layers, that means we want 4m triangles, so $n = 2m$. But we find that the last extra bit of the $4m^{th}$ triangle is not needed in the last layer. So, after a detailed look one finds that for m layers, you need a_1, 4m-1 big triangles after that, and the first half of the $4m^{th}$ big triangle. Basically the length of the strip for m layers is :

$$L(mlayers) = 2(a\tan\theta + a\tan^3\theta + ... + a\tan^{4m-1}\theta) \tag{8}$$
$$= 2a\tan\theta(1 + \tan^2\theta + ... + \tan^{2(2m-1)}\theta) \tag{9}$$
$$= 2a\tan\theta \frac{(1 - (\tan^2\theta)^{2m})}{1 - \tan^2\theta} \tag{10}$$

For the geometry of this strip,

$$\sin 2\theta = \frac{a - b}{\sqrt{(a-b)^2 + L^2}} \tag{11}$$
$$\cos 2\theta = \frac{L}{\sqrt{(a-b)^2 + L^2}} \tag{12}$$
$$\tan\theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} \tag{13}$$

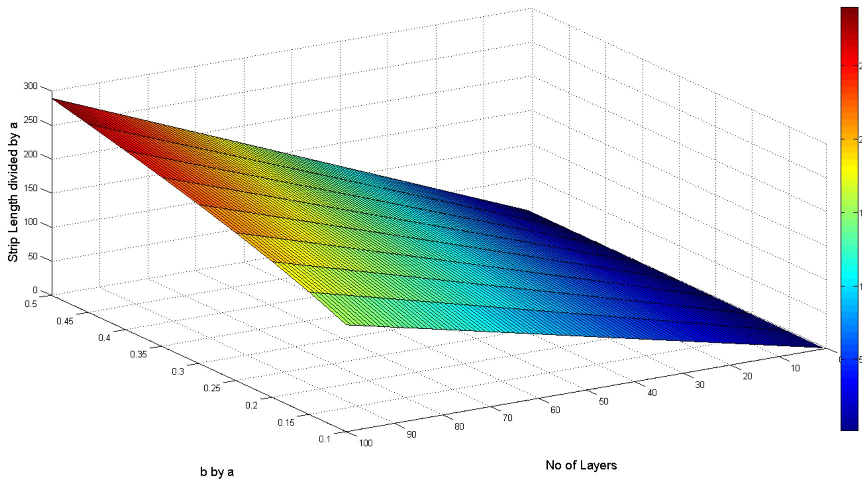

Figure 4: A surface plot showing how the strip length depends on number of layers and the relative size of the big and small squares. Here $\beta = b/a$ and $\lambda = L/a$ where a,b and L are as defined in Fig. 3

If we make the outer dimension of the square, a, an independent parameter, then we can work in terms of quantities $\lambda = \frac{L}{a}$ and $\beta = \frac{b}{a}$.

Then,

$$\lambda(4m) = 2\tan\theta \frac{(1-\tan^{4m}\theta)}{1-\tan^2\theta} \tag{14}$$

And similarly $\tan\theta$ will just depend on λ and β.

So, in principle we should be able to substitute $\tan\theta$ from eq. 13 in eq. 14 and get a relation between λ, m and β.

I haven't been able to solve it analytically, especially since it involves unknown powers. There could be a way to solve it analytically, using some approximations and/or logarithms, but I havent explored those yet. For the interested reader, the most interesting mathematical property of this origami is it being a fractal - which basically means that it contains a structure repeating inside itself.

I did a numerical simulation of this instead, and found that it was easy to find a converging result by starting at some value of $\tan\theta$ and then using that to find the corresponding λ, use that λ to get the next $\tan\theta$ and so on until it converges.

I plotted the required strip length λ for various values of m and β in Fig 4 One can also look at the dependence of λ on β and m one at a time, as shown in Fig 5 and 6.

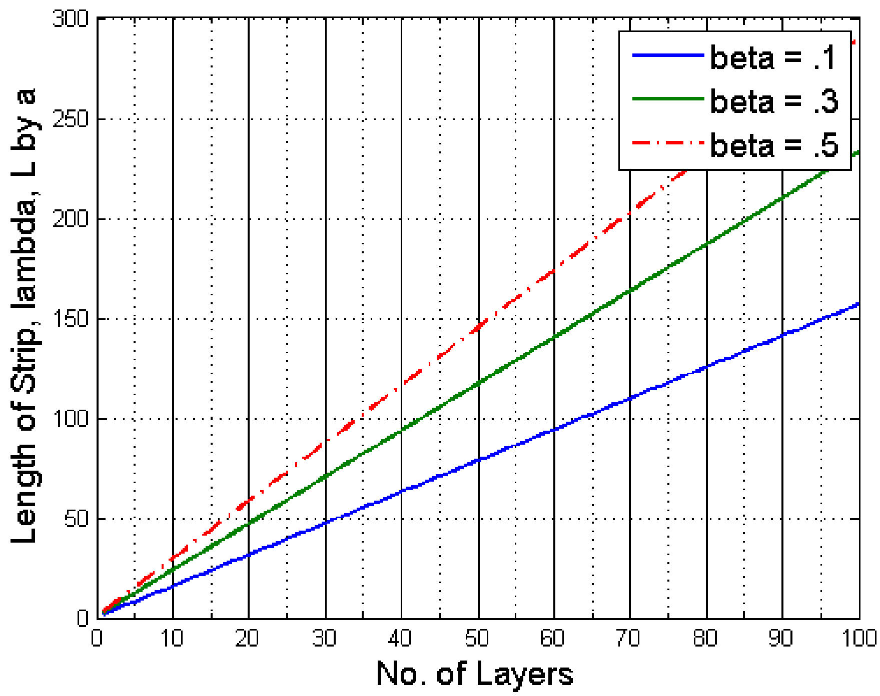

Figure 5: Dependence of the Length of Strip required on no. of layers of the curlicue, given a fixed size of the squares.

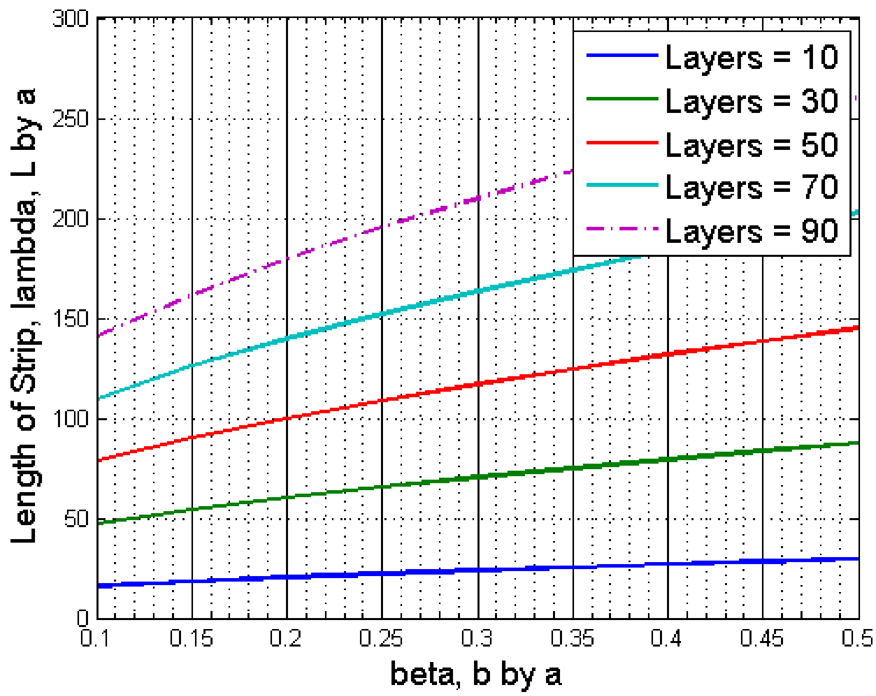

Figure 6: Dependence of strip length on the size of the small square, given a fixed number of layers.